Problem 4.29

(a) Check that the spin matrices (Equations 4.145 and 4.147) obey the fundamental commutation relations for angular momentum, Equation 4.134.

(b) Show that the Pauli spin matrices (Equation 4.148) satisfy the product rule

$$\sigma_j\sigma_k = \delta_{jk} + i\sum_l \epsilon_{jkl}\sigma_l, \tag{4.153}$$

where the indices stand for x, y, or z, and ϵ_{jkl} is the **Levi-Civita** symbol: $+1$ if $jkl = 123$, 231, or 312; -1 if $jkl = 132$, 213, or 321; 0 otherwise.

[TYPO: δ_{jk} is a scalar, whereas the other terms in the equation are 2×2 matrices. Replace it with $\delta_{jk}\mathsf{I}$, the 2×2 matrix whose entries are δ_{jk} on the main diagonal and zero everywhere else.]

Solution

Part (a)

The spin matrices are in Equations 4.145 and 4.147 on page 168.

$$\mathsf{S}_x = \frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \qquad \mathsf{S}_y = \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} \qquad \mathsf{S}_z = \frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

The fundamental commutation relations for spin angular momentum are in Equation 4.134 on page 166.

$$[S_x, S_y] = i\hbar S_z \qquad [S_y, S_z] = i\hbar S_x \qquad [S_z, S_x] = i\hbar S_y$$

Verify the first commutation relation.

$$\begin{aligned}
[S_x, S_y] = S_xS_y - S_yS_x &= \frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} - \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\\
&= \frac{\hbar^2}{4}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} - \frac{\hbar^2}{4}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\\
&= \frac{\hbar^2}{4}\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix} - \frac{\hbar^2}{4}\begin{bmatrix}-i & 0\\ 0 & i\end{bmatrix}\\
&= \frac{\hbar^2}{4}\left(\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix} - \begin{bmatrix}-i & 0\\ 0 & i\end{bmatrix}\right)\\
&= \frac{\hbar^2}{4}\begin{bmatrix}2i & 0\\ 0 & -2i\end{bmatrix}\\
&= i\hbar\left(\frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right)\\
&= i\hbar S_z
\end{aligned}$$

Verify the second commutation relation.

$$\begin{aligned}
[S_y, S_z] &= S_y S_z - S_z S_y \\
&= \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= \frac{\hbar^2}{4}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \frac{\hbar^2}{4}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= \frac{\hbar^2}{4}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \frac{\hbar^2}{4}\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \\
&= \frac{\hbar^2}{4}\left(\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}\right) \\
&= \frac{\hbar^2}{4}\begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} \\
&= i\hbar\left(\frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \\
&= i\hbar S_x
\end{aligned}$$

Verify the third commutation relation.

$$\begin{aligned}
[S_z, S_x] &= S_z S_x - S_x S_z \\
&= \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \frac{\hbar^2}{4}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \frac{\hbar^2}{4}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \frac{\hbar^2}{4}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \frac{\hbar^2}{4}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
&= \frac{\hbar^2}{4}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) \\
&= \frac{\hbar^2}{4}\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \\
&= i\hbar\left(\frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\right) \\
&= i\hbar S_y
\end{aligned}$$

Part (b)

The Pauli matrices are in Equation 4.148 on page 168.

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Start by checking that they do in fact satisfy

$$\sigma_j \sigma_k = \delta_{jk} \mathsf{I} + i \sum_{l=1}^{3} \varepsilon_{jkl} \sigma_l$$

by evaluating the left side and the right side separately for each possible product.

$$\begin{cases} \sigma_x \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \delta_{11} \mathsf{I} + i \sum_{l=1}^{3} \varepsilon_{11l} \sigma_l = 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i(\varepsilon_{111}\sigma_x + \varepsilon_{112}\sigma_y + \varepsilon_{113}\sigma_z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{cases}$$

$$\begin{cases} \sigma_x \sigma_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \\ \delta_{12} \mathsf{I} + i \sum_{l=1}^{3} \varepsilon_{12l} \sigma_l = 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i(\varepsilon_{121}\sigma_x + \varepsilon_{122}\sigma_y + \varepsilon_{123}\sigma_z) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \end{cases}$$

$$\begin{cases} \sigma_x \sigma_z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ \delta_{13} \mathsf{I} + i \sum_{l=1}^{3} \varepsilon_{13l} \sigma_l = 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i(\varepsilon_{131}\sigma_x + \varepsilon_{132}\sigma_y + \varepsilon_{133}\sigma_z) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{cases}$$

$$\begin{cases} \sigma_y \sigma_x = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\ \delta_{21} \mathsf{I} + i \sum_{l=1}^{3} \varepsilon_{21l} \sigma_l = 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i(\varepsilon_{211}\sigma_x + \varepsilon_{212}\sigma_y + \varepsilon_{213}\sigma_z) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \end{cases}$$

$$\begin{cases} \sigma_y \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \delta_{22} \mathsf{I} + i \sum_{l=1}^{3} \varepsilon_{22l} \sigma_l = 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i(\varepsilon_{221}\sigma_x + \varepsilon_{222}\sigma_y + \varepsilon_{223}\sigma_z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{cases}$$

Evaluate the left and right sides for the remaining products.

$$\begin{cases} \sigma_y\sigma_z = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \\ \delta_{23}\mathsf{I} + i\sum_{l=1}^{3} \varepsilon_{23l}\sigma_l = 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i(\varepsilon_{231}\sigma_x + \varepsilon_{232}\sigma_y + \varepsilon_{233}\sigma_z) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \end{cases}$$

$$\begin{cases} \sigma_z\sigma_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ \delta_{31}\mathsf{I} + i\sum_{l=1}^{3} \varepsilon_{31l}\sigma_l = 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i(\varepsilon_{311}\sigma_x + \varepsilon_{312}\sigma_y + \varepsilon_{313}\sigma_z) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{cases}$$

$$\begin{cases} \sigma_z\sigma_y = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \\ \delta_{32}\mathsf{I} + i\sum_{l=1}^{3} \varepsilon_{32l}\sigma_l = 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i(\varepsilon_{321}\sigma_x + \varepsilon_{322}\sigma_y + \varepsilon_{323}\sigma_z) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \end{cases}$$

$$\begin{cases} \sigma_z\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \delta_{33}\mathsf{I} + i\sum_{l=1}^{3} \varepsilon_{33l}\sigma_l = 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i(\varepsilon_{331}\sigma_x + \varepsilon_{332}\sigma_y + \varepsilon_{333}\sigma_z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{cases}$$

Now prove the result by writing a general formula for the Pauli matrices,

$$\sigma_j = \begin{bmatrix} \delta_{3j} & \delta_{1j} - i\delta_{2j} \\ \delta_{1j} + i\delta_{2j} & -\delta_{3j} \end{bmatrix},$$

and evaluating the product of $\sigma_j\sigma_k$.

$$\begin{aligned} \sigma_j\sigma_k &= \begin{bmatrix} \delta_{3j} & \delta_{1j} - i\delta_{2j} \\ \delta_{1j} + i\delta_{2j} & -\delta_{3j} \end{bmatrix} \begin{bmatrix} \delta_{3k} & \delta_{1k} - i\delta_{2k} \\ \delta_{1k} + i\delta_{2k} & -\delta_{3k} \end{bmatrix} \\ &= \begin{bmatrix} \delta_{3j}\delta_{3k} + (\delta_{1j} - i\delta_{2j})(\delta_{1k} + i\delta_{2k}) & \delta_{3j}(\delta_{1k} - i\delta_{2k}) + (\delta_{1j} - i\delta_{2j})(-\delta_{3k}) \\ (\delta_{1j} + i\delta_{2j})\delta_{3k} + (-\delta_{3j})(\delta_{1k} + i\delta_{2k}) & (\delta_{1j} + i\delta_{2j})(\delta_{1k} - i\delta_{2k}) + (-\delta_{3j})(-\delta_{3k}) \end{bmatrix} \\ &= \begin{bmatrix} (\delta_{1j}\delta_{1k} + \delta_{2j}\delta_{2k} + \delta_{3j}\delta_{3k}) + i(\delta_{1j}\delta_{2k} - \delta_{2j}\delta_{1k}) & (\delta_{3j}\delta_{1k} - \delta_{1j}\delta_{3k}) + i(\delta_{2j}\delta_{3k} - \delta_{3j}\delta_{2k}) \\ (\delta_{1j}\delta_{3k} - \delta_{3j}\delta_{1k}) + i(\delta_{2j}\delta_{3k} - \delta_{3j}\delta_{2k}) & (\delta_{1j}\delta_{1k} + \delta_{2j}\delta_{2k} + \delta_{3j}\delta_{3k}) + i(\delta_{2j}\delta_{1k} - \delta_{1j}\delta_{2k}) \end{bmatrix} \end{aligned}$$

Recognize that $\delta_{1j}\delta_{1k} + \delta_{2j}\delta_{2k} + \delta_{3j}\delta_{3k}$ is 1 only if j and k are equal and 0 otherwise.

$$\sigma_j\sigma_k = \begin{bmatrix} \delta_{jk} + i(\delta_{1j}\delta_{2k} - \delta_{2j}\delta_{1k}) & (\delta_{3j}\delta_{1k} - \delta_{1j}\delta_{3k}) + i(\delta_{2j}\delta_{3k} - \delta_{3j}\delta_{2k}) \\ (\delta_{1j}\delta_{3k} - \delta_{3j}\delta_{1k}) + i(\delta_{2j}\delta_{3k} - \delta_{3j}\delta_{2k}) & \delta_{jk} + i(\delta_{2j}\delta_{1k} - \delta_{1j}\delta_{2k}) \end{bmatrix}$$

$$= \begin{bmatrix} \delta_{jk} & 0 \\ 0 & \delta_{jk} \end{bmatrix} + \begin{bmatrix} i(\delta_{1j}\delta_{2k} - \delta_{2j}\delta_{1k}) & -(\delta_{1j}\delta_{3k} - \delta_{3j}\delta_{1k}) + i(\delta_{2j}\delta_{3k} - \delta_{3j}\delta_{2k}) \\ (\delta_{1j}\delta_{3k} - \delta_{3j}\delta_{1k}) + i(\delta_{2j}\delta_{3k} - \delta_{3j}\delta_{2k}) & -i(\delta_{1j}\delta_{2k} - \delta_{2j}\delta_{1k}) \end{bmatrix}$$

Using the fact that

$$\sum_{l=1}^{3} \varepsilon_{mnl}\varepsilon_{jkl} = \delta_{mj}\delta_{nk} - \delta_{nj}\delta_{mk}$$

this matrix product becomes

$$\sigma_j\sigma_k = \delta_{jk}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} i\sum_{l=1}^{3}\varepsilon_{12l}\varepsilon_{jkl} & -\sum_{l=1}^{3}\varepsilon_{13l}\varepsilon_{jkl} + i\sum_{l=1}^{3}\varepsilon_{23l}\varepsilon_{jkl} \\ \sum_{l=1}^{3}\varepsilon_{13l}\varepsilon_{jkl} + i\sum_{l=1}^{3}\varepsilon_{23l}\varepsilon_{jkl} & -i\sum_{l=1}^{3}\varepsilon_{12l}\varepsilon_{jkl} \end{bmatrix}$$

$$= \delta_{jk}\mathsf{I} + \begin{bmatrix} i(\varepsilon_{121}\varepsilon_{jk1} + \varepsilon_{122}\varepsilon_{jk2} + \varepsilon_{123}\varepsilon_{jk3}) & -(\varepsilon_{131}\varepsilon_{jk1} + \varepsilon_{132}\varepsilon_{jk2} + \varepsilon_{133}\varepsilon_{jk3}) + i\sum_{l=1}^{3}\varepsilon_{23l}\varepsilon_{jkl} \\ (\varepsilon_{131}\varepsilon_{jk1} + \varepsilon_{132}\varepsilon_{jk2} + \varepsilon_{133}\varepsilon_{jk3}) + i\sum_{l=1}^{3}\varepsilon_{23l}\varepsilon_{jkl} & -i(\varepsilon_{121}\varepsilon_{jk1} + \varepsilon_{122}\varepsilon_{jk2} + \varepsilon_{123}\varepsilon_{jk3}) \end{bmatrix}$$

$$= \delta_{jk}\mathsf{I} + \begin{bmatrix} i\varepsilon_{jk3} & -(-\varepsilon_{jk2}) + i\sum_{l=1}^{3}\varepsilon_{23l}\varepsilon_{jkl} \\ -\varepsilon_{jk2} + i\sum_{l=1}^{3}\varepsilon_{23l}\varepsilon_{jkl} & -i\varepsilon_{jk3} \end{bmatrix}$$

$$= \delta_{jk}\mathsf{I} + \begin{bmatrix} i\varepsilon_{jk3} & \varepsilon_{jk2} + i(\varepsilon_{231}\varepsilon_{jk1} + \varepsilon_{232}\varepsilon_{jk2} + \varepsilon_{233}\varepsilon_{jk3}) \\ -\varepsilon_{jk2} + i(\varepsilon_{231}\varepsilon_{jk1} + \varepsilon_{232}\varepsilon_{jk2} + \varepsilon_{233}\varepsilon_{jk3}) & -i\varepsilon_{jk3} \end{bmatrix}$$

$$= \delta_{jk}\mathsf{I} + \begin{bmatrix} i\varepsilon_{jk3} & \varepsilon_{jk2} + i\varepsilon_{jk1} \\ -\varepsilon_{jk2} + i\varepsilon_{jk1} & -i\varepsilon_{jk3} \end{bmatrix}$$

$$= \delta_{jk}\mathsf{I} + i\begin{bmatrix} \varepsilon_{jk3} & -i\varepsilon_{jk2} + \varepsilon_{jk1} \\ i\varepsilon_{jk2} + \varepsilon_{jk1} & -\varepsilon_{jk3} \end{bmatrix}.$$

Therefore,

$$\begin{aligned}
\sigma_j\sigma_k &= \delta_{jk}\mathsf{I} + i\left(\varepsilon_{jk1}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + \varepsilon_{jk2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} + \varepsilon_{jk3}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right) \\
&= \delta_{jk}\mathsf{I} + i(\varepsilon_{jk1}\sigma_x + \varepsilon_{jk2}\sigma_y + \varepsilon_{jk3}\sigma_z) \\
&= \delta_{jk}\mathsf{I} + i\sum_{l=1}^{3}\varepsilon_{jkl}\sigma_l.
\end{aligned}$$